

Integral Calculus

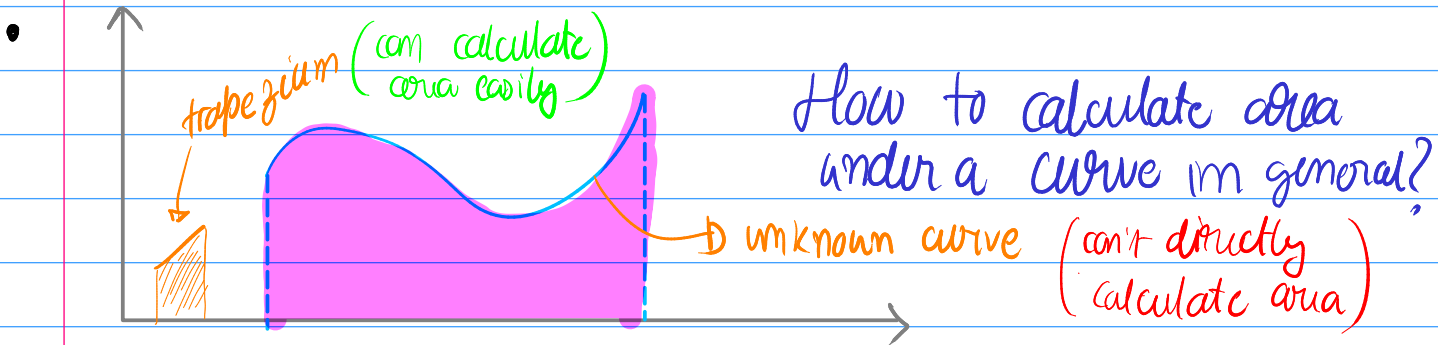
ISI & CMI Crash Course 2024

lec 1

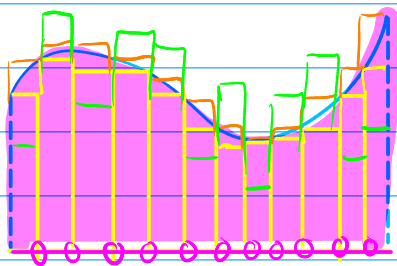
2024-04-25

★ Riemann Integrals.

⊙ Motivation



Importantly, we know how to calculate areas of trapeziums (or more simply, rectangles)



So, approximate unknown area using rectangles!

Now, which rectangles to pick?

↳ why equal width rectangles?? Could've done something else.

We can intuitively see that as the number of rectangles increases, our approximation should improve, but there is a lot of freedom in how these rectangles are chosen, and it isn't clear that the approximation works regardless of what choices are made, given some basic assumptions, of course.

We shall spend the next ~~few lectures~~ formalising & refining this idea.
nope, all today.

○ Partitions

- The first thing to do is to slice up the interval $I = [a, b]$ into smaller "disjoint" intervals. This is achieved through a partition.

- Defn: A partition of a closed, bounded (viz compact) interval $I = [a, b] \subseteq \mathbb{R}$ is a tuple

$$\mathcal{P} = (x_0, x_1, x_2, \dots, x_{n-1}, x_n)$$

where $n \in \mathbb{N}$, $a = x_0 < x_1 < \dots < x_n = b$; $x_i \in \mathbb{R} \forall i$

The points $\{x_i\}$ are called segmentation points / cut points

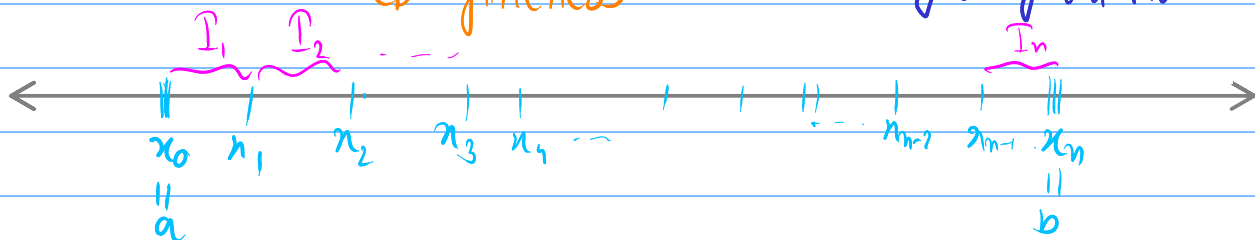
- Defn: $I_j := [x_{j-1}, x_j]$; $j = 1, \dots, n$

segments / sub-intervals.

$$l(I_j) := x_j - x_{j-1} \quad ; \quad j = 1, \dots, n$$

$$\|\mathcal{P}\| = \max_{1 \leq j \leq n} l(I_j) \leftarrow \begin{array}{l} \text{"norm" or "mesh"} \\ \text{of a partition} \end{array}$$

(p "fineness")



Hence, $I_j \cap I_{j+1} = \{x_j\}$; $\bigcup_{j=1}^n I_j = I = [a, b]$

(now, have to fix heights of each rectangle...)

- max or min
 - take on endpoint
 - take any value x (bad)
 - take value @ any point
- special case of I
- we don't use f at all!

◉ Idea I: Take value of f @ any point $\rightarrow$ "ta"

- Defⁿ: A tagged partition $\dot{\mathcal{P}}$ can be defined from $\mathcal{P} = (x_0=a, \dots, x_n=b)$ as follows.

$$\dot{\mathcal{P}} = \{(\mathcal{I}_j, t_j)\}_{j=1}^n \leftarrow \text{set of 2-tuples.}$$

where $t_j \in \mathcal{I}_j$. $\dot{\mathcal{P}}$ $\leftarrow$ signifies that the partition (arbitrarily) is tagged.

- Defⁿ: $\|\dot{\mathcal{P}}\| = \|\mathcal{P}\|$ $\hookrightarrow$ ht of rectangles $\sim \mathcal{I}_j$

$$S(f; \dot{\mathcal{P}}) = \sum_{j=1}^n \widetilde{f(t_j)} l(\mathcal{I}_j) = \sum_{j=1}^n f(t_j)(x_j - x_{j-1})$$

$\hookrightarrow$ This is called the Riemann sum of $\dot{\mathcal{P}}$ wrt f

◉ Idea II: Take the maximum & minimum value of f in segment. (tagging not necessary).

- Defⁿ: $m_j = \inf_{x \in \mathcal{I}_j} f(x)$ $M_j = \sup_{x \in \mathcal{I}_j} f(x)$

- Defⁿ: $L(f; \mathcal{P}) = \sum_{j=1}^n m_j l(\mathcal{I}_j) = \sum_{j=1}^n m_j (x_j - x_{j-1})$

$\hookrightarrow$ Lower Darboux summation.

$$U(f; \mathcal{P}) = \sum_{j=1}^n M_j l(\mathcal{I}_j) = \sum_{j=1}^n M_j (x_j - x_{j-1})$$

$\hookrightarrow$ Upper Darboux summation.

⊙ Riemann Integrability

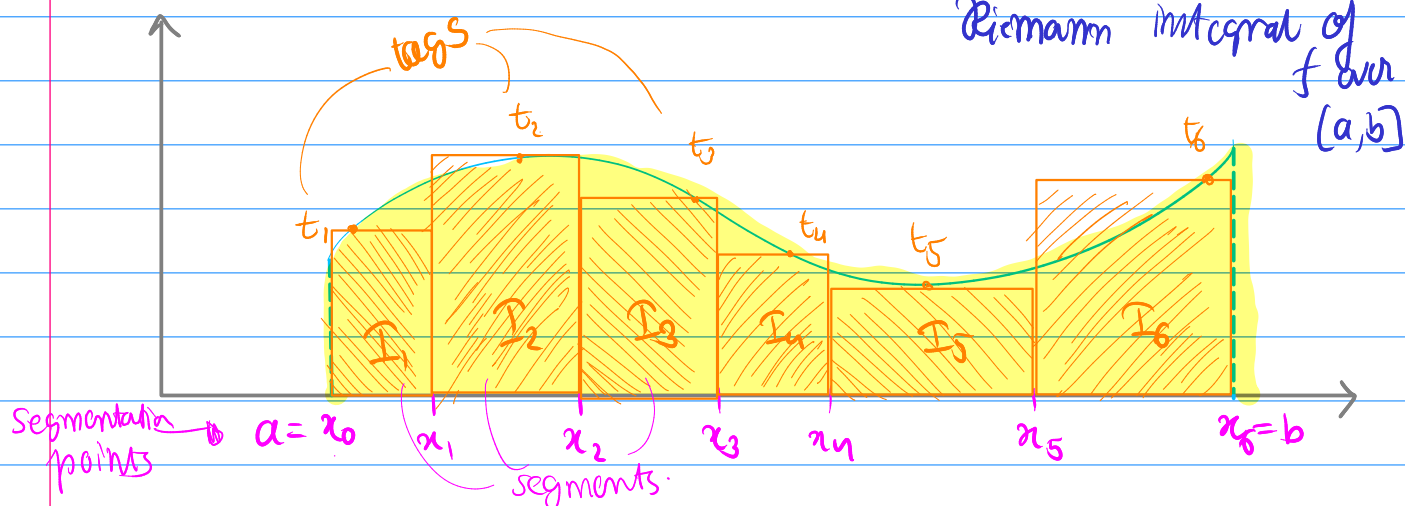
- Defn: We say f is Riemann integrable on $[a, b]$ ($f \in \mathcal{R}[a, b]$) iff $\exists L \in \mathbb{R} \mid \forall \varepsilon > 0 \exists \delta > 0$

$$\|\dot{P}\| < \delta \Rightarrow |S(f; \dot{P}) - L| < \varepsilon$$

ignore { Equivalently, for a sequence of successive refinements $\{\dot{P}_n\}$

$$\|\dot{P}_n\| \rightarrow 0 \Rightarrow S(f; \dot{P}_n) \rightarrow L$$

 (arbitrarily tagged. & L is called the Riemann integral of f over $[a, b]$)



$$P = (x_0, x_1, x_2, x_3, x_4, x_5, x_6)$$

$$\dot{P} = \left\{ \begin{array}{c} (I_1, t_1), (I_2, t_2), \dots, (I_6, t_6) \\ \parallel \qquad \qquad \qquad \parallel \\ [x_0, x_1] \qquad \qquad \qquad [x_5, x_6] \end{array} \right\}$$

- Theorem: The Riemann integral of f (where $f \in \mathcal{R}[a, b]$) over $[a, b]$ is unique.
 $\therefore$ it is unique, it is denoted as: $\int_a^b f(x) dx$ where x is the variable of integration
 ... or simply: $\int_a^b f$

◉ An Alternative Approach: Darboux Summations.

- We have already defined:

$$U(f; \mathcal{P}) = \sum_j M_j \ell(I_j) \text{ where } M_j = \sup_{x \in I_j} f(x)$$

$$L(f; \mathcal{P}) = \sum_j m_j \ell(I_j) \text{ where } m_j = \inf_{x \in I_j} f(x)$$

- We define the upper Riemann integral of f over $[a, b]$ as:

$$\int_a^b f(x) dx = \int_a^b f = \inf_{\mathcal{P}} U(f; \mathcal{P})$$

- & define the lower Riemann integral of f over $[a, b]$ as:

$$\int_a^b f(x) dx = \int_a^b f = \sup_{\mathcal{P}} L(f; \mathcal{P})$$

- Def'n: We say f is Riemann integrable over $[a, b]$ ($f \in \mathcal{R}[a, b]$) iff:

$$\int_a^b f(x) dx = \int_a^b f(x) dx$$

... and the common value is called the Riemann integral of f over $[a, b]$ and denoted as: $\int_a^b f(x) dx$

or simply: $\int_a^b f$.

variable of integration.

- Lemma: If $\mathcal{P}'$ refines $\mathcal{P}$, $U(f; \mathcal{P}') \leq U(f; \mathcal{P})$
 $L(f; \mathcal{P}') \geq L(f; \mathcal{P})$

Proof: HW.

- Lemma: For a tagged partition $\dot{\mathcal{P}}$,

$$L(f; \mathcal{P}) \leq S(f; \dot{\mathcal{P}}) \leq U(f; \mathcal{P})$$

Proof: HW (hint: $\forall j \quad m_j \leq f(t_j) \leq M_j$)

- Equivalence Theorem:
 Defn 1 $\leftarrow$ Riemann sum def'n
 Defn 2 $\leftarrow$ Darboux sum def'n.

The two definitions of Riemann integrability are equivalent.

- Theorem f is Riemann integrable on $[a, b]$ iff
 $\forall \varepsilon > 0 \exists \mathcal{P} \text{ on } [a, b] \mid U(f; \mathcal{P}) - L(f; \mathcal{P}) < \varepsilon$

- Boundedness Theorem ($\S 7.1.6$ of Bartle-Shurport)

$$f \in \mathcal{R}[a, b] \Rightarrow f \text{ is bounded on } [a, b]$$

Idea is: If f is unbounded on $[a, b]$, for any $\mathcal{P}$,
 $\exists j \mid f$ is unbounded on $I_j \Rightarrow$ can choose $t_j \mid f(t_j)$
 explodes.

② Algebra of Riemann Integrals

- If $f, g \in \mathcal{R}[a, b]$, $\lambda \in \mathbb{R}$, then $f + \lambda g \in \mathcal{R}[a, b]$
and:

$$\int_a^b (f + \lambda g) = \int_a^b f + \lambda \int_a^b g$$

- If $f, g \in \mathcal{R}[a, b]$, $\forall x \in [a, b]$ $f(x) \leq g(x)$

then:

$$\int_a^b f(x) \leq \int_a^b g(x)$$

- (Squeeze / Sandwich Theorem works, thanks to ↑)

- Additivity Theorem: $f \in \mathcal{R}[a, b] \Leftrightarrow \forall c \in (a, b) \ f \in \mathcal{R}[a, c] \cap \mathcal{R}[c, b]$

and:

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$$

where $a < c < b$.

(Proof is too advanced)

- If $f \in \mathcal{R}[a, b]$ then $f \in \mathcal{R}[c, d] \ \forall \ [c, d] \subseteq [a, b]$

- Convention: $\int_a^a (\dots) = 0$ & $\int_a^b (\dots) = -\int_b^a (\dots)$

- King's Rule: $\int_a^b f(x) dx = \int_a^b f(a+b-x) dx$

⊙ Classes of Riemann Integrable Functions (§ 7.2 of Bartle-Shurwat)

- Result: Constant functions are Riemann integrable.

Proof: $\forall P, \forall j \quad m_j = M_j \Rightarrow U(f; P) = L(f; P)$

- Result: All step functions (simple f's) are Riemann integrable

... take finite number of discrete values over a bounded set

Hint: First consider elementary step functions:

1) ($x \in \text{interval}$)

Then, show that all step f's are linear combinations of elementary step functions.

$$f(x) = \begin{cases} 1 & \text{in } [1, 2] \\ 2 & \text{in } (2, 3] \\ 3 & \text{in } (3, 4] \\ 0 & \text{o/w} \end{cases} = 1 \cdot \mathbb{1}_{[1, 2]} + 2 \cdot \mathbb{1}_{(2, 3]} + 3 \cdot \mathbb{1}_{(3, 4]}$$

- Result: All continuous functions are integrable. (HW)

- Result: All monotone functions are integrable.

Proof is hard

- Lebesgue Integrability Criterion

A function is Riemann integrable iff it is continuous almost everywhere.

Also: $\int_A \dots = 0$
 $A \rightarrow \text{measure zero}$

↳ i.e. ... except for a measure zero set.

countable / finite $\Rightarrow$ measure zero.

↳ "can be covered by a set of intervals w/ arbitrarily small combined length"

(a) i) $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \sin\left(\frac{k}{n}\pi\right)$

ii) $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^n e^{k/n}$

iii) $\lim_{n \rightarrow \infty} \frac{1}{n} (n!)^{1/n}$

iv) $\lim_{n \rightarrow \infty} \frac{1^\alpha + 2^\alpha + \dots + n^\alpha}{n^{\alpha+1}}$

• Result: $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n f\left(\frac{k}{n}\right) = \int_0^1 f(x) dx$ if $f \in R[0,1]$

Proof: Consider $[0,1]$ partitioned as:

$$\mathcal{P}_n = \left(0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, 1\right) \Rightarrow \|\mathcal{P}_n\| = \frac{1}{n} \equiv \ell(\mathcal{I})$$

Let tags be the right endpoints:

$$\dot{\mathcal{P}}_n = \left\{ \left(\left[\frac{j-1}{n}, \frac{j}{n} \right], \frac{j}{n} \right) \right\}_{j=1}^n$$

$$S(f; \dot{\mathcal{P}}_n) = \sum_{j=1}^n f(c_j) \ell(\mathcal{I}_j) = \sum_{j=1}^n f\left(\frac{j}{n}\right) \frac{1}{n}$$

Now, take $\lim_{n \rightarrow \infty} S(f; \dot{\mathcal{P}}_n) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n f\left(\frac{j}{n}\right)$

RHS = $\int_0^1 f(x) dx$; if $f \in R[0,1]$ LHS.

Ans. i) $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \sin\left(\frac{k}{n} \pi\right) = \int_0^1 \sin(xt) dt$

$$= \frac{1}{\pi} \int_0^{\pi} \sin(u) du \quad [u := xt \Rightarrow du = x dt]$$

$$= \frac{1}{\pi} \left[-\cos(u) \right]_{u=0}^{u=\pi}$$

$$= \frac{1 - \cos \pi}{\pi}$$

$$\lim_{n \rightarrow \infty} \frac{1}{\pi} \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \sin\left(\frac{k\pi}{n}\right) = \lim_{n \rightarrow \infty} \frac{1 - \cos \pi}{\pi^2} = \frac{1}{2}$$

ii) $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^n e^{-k/n} = \lim_{n \rightarrow \infty} \frac{1}{n} + \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n e^{-k/n}$

$$= \int_0^1 e^{-t} dt = -e^{-t} \Big|_0^1 = -e^{-1} - (-1) = 1 - \frac{1}{e}$$

$$\approx 63.2\%$$

iii) $\lim_{n \rightarrow \infty} \frac{1}{n} (n!)^{1/n} = \lim_{n \rightarrow \infty} \left(\frac{n!}{n^n} \right)^{1/n}$

$$= \lim_{n \rightarrow \infty} \left(\prod_{k=1}^n \frac{k}{n} \right)^{1/n} \quad \text{Taking logarithm:}$$

$$\ln L = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \ln\left(\frac{k}{n}\right) = \int_0^1 \ln(x) dx = \int_{-\infty}^0 e^y dy$$

$$= e^y \Big|_{-\infty}^0 = 1$$

(by reflection of graph)

Errata 2024-04-28

$nPr = k!$, $nC_k = \frac{n!}{(n-k)!} = \frac{n!}{\prod_{i=1}^k (n-i+1)}$ = # ways to pick k balls in order from n distinct balls.

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left[{}^{2n}P_n \right]^{1/n}$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \left[\frac{(2n)!}{(n!)^2} \right]^{1/n}$$

$$= \lim_{n \rightarrow \infty} \left[\frac{2n \cdot (2n-1) \cdot \dots \cdot (n+1) \cdot \cancel{n!}}{n^n \cancel{n!}} \right]^{1/n}$$

$$= \lim_{n \rightarrow \infty} \left[\prod_{k=1}^n \left(\frac{n+k}{n} \right) \right]^{1/n} \cdot \text{Taking ln, a cls. f}^n!$$

$$\therefore \ln L = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \ln \left(1 + \frac{k}{n} \right) = \int_0^1 \ln(1+x) dx$$

$$= \int_1^2 \ln(t) dt = \left[t \ln(t) - t \right]_1^2 \quad (\text{by parts}) \quad \text{H.W.}$$

$$= 2 \ln 2 - 2 - (-1) + 1 = \underline{\underline{2 \ln 2 - 1}} \approx 0.386$$

iv) $\lim_{n \rightarrow \infty} \frac{1^\alpha + 2^\alpha + \dots + n^\alpha}{n^{\alpha+1}}$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \left(\frac{k}{n} \right)^\alpha = \int_0^1 x^\alpha dx = \frac{1}{\alpha+1}$$

⊛ Fundamental Theorem of Calculus (§7.3 Bartle-Shorbert)

⊙ First Form

• Theorem: Suppose $\exists E \subseteq [a, b]$ finite and
 $f, F: [a, b] \rightarrow \mathbb{R}$ s.t.

- i) F is continuous on $[a, b]$
- ii) $F'(x) = f(x)$ on $[a, b] \setminus E$
- iii) $f \in \mathcal{R}[a, b]$

$$\int_a^b f(x) dx = F(b) - F(a)$$

Proof: w/o loss of generality, assume $E = \{a, b\}$.
 If not, we can break ---

$$\int_a^b f(x) dx = \int_a^{c_1} f(x) dx + \int_{c_1}^{c_2} f(x) dx + \dots + \int_{c_k}^b f(x) dx$$

--- then prove for each part separately. (telescope)

Fix any $\varepsilon > 0$. $\because f \in \mathcal{R}[a, b] \exists \delta > 0 \mid$

$$\forall \mathcal{P} \mid \|\mathcal{P}\| < \delta, \quad \left| S(f; \mathcal{P}) - \int_a^b f \right| < \varepsilon$$

Pick any $\mathcal{P} \mid \|\mathcal{P}\| < \delta$.

We tag $\mathcal{P}$ to get $\check{\mathcal{P}} \mid S(f; \check{\mathcal{P}}) = F(b) - F(a)$
 (Claim 1)

Supposing the above claim holds, we have

$$\left| F(b) - F(a) - \int_a^b f \right| < \varepsilon$$

This holds $\forall \varepsilon$, as we can always get a corresp δ , and hence a $\mathcal{P}$ s.t. $\|\mathcal{P}\| < \delta$.

By uniqueness of Riemann integral,

$$\int_a^b f(x) dx = F(b) - F(a). \quad \square$$

Proof of Claim 1:

Let $\mathcal{P} = (a = x_0, x_1, \dots, x_{n-1}, x_n = b)$

$$\therefore F(b) - F(a) = \sum_{j=1}^n F(x_j) - F(x_{j-1})$$

$$= \sum_{j=1}^n F'(u_j) (x_j - x_{j-1}) \quad ; \quad u_j \in I_j^\circ = (x_{j-1}, x_j) \quad [\text{by LMVT}]$$

$$= \sum_{j=1}^n f(u_j) \ell(I_j) \quad (\because u_j \in E)$$

$$\text{Let } \dot{\mathcal{P}} = \left\{ (I_j, u_j) \mid u_j \in I_j^\circ \right\}_{j=1}^n$$

$$\therefore S(f; \dot{\mathcal{P}}) = F(b) - F(a) \quad \square$$

⊙ Second Form

• Defⁿ Let $f, F : [a, b] \rightarrow \mathbb{R}$

i) F is continuous on $[a, b]$

ii) F is differentiable on (a, b) and $\forall x \in (a, b) \ F'(x) = f(x)$

Then F is called an antiderivative of f .

aka
(primitive)

• Result: Let $f, F : [a, b] \rightarrow \mathbb{R}$, F is the antiderivative of f . Then $F' : [a, b] \rightarrow \mathbb{R}$ is an antiderivative of f iff $(F - F')$ is a constant f^n on $[a, b]$

How

• Theorem Let $f, F : [a, b] \rightarrow \mathbb{R}$, F is the antiderivative of f . If $f \in \mathcal{R}[a, b]$, then

$$\int_a^b f(x) dx = F(b) - F(a)$$

Proof: (follows from the First Form)

• Defⁿ: If $f \in \mathcal{R}[a, b]$, then the f^n :

$$F(z) = \int_a^z f(x) dx \quad ; \quad z \in [a, b]$$

... is called the indefinite integral of f with basepoint a .

• Result: $f \in \mathcal{R}[a, b]$ - $F(z) = \int_a^z f(x) dx$

Then F is Lipschitz on $[a, b]$, hence ds.

(Proof follows from boundedness of f)

- Result! $f \in \mathcal{R}[a, b]$, $F(x) := \int_a^x f(t) dt$, $x \in [a, b]$

f is continuous @ $c \Rightarrow F$ is differentiable @ c ,
 $F'(c) = f(c)$

(Boole-Stieltjes has proof)

- $\therefore$ If $f \in \mathcal{R}[a, b]$, $F(x) := \int_a^x f(t) dt$ is an antiderivative* of f .

* except @ points of discontinuity of f .

NOTE: $f \in \mathcal{R}[a, b] \Leftrightarrow f$ is cts. almost everywhere on $[a, b]$.

$\therefore$ FTC's second form holds for almost all points in $[a, b]$

- Commonly, the indefinite integral of f is denoted by:
 $\int f$ (w/o bounds) $\leftarrow$ which are integrable on $\mathbb{R}$
 $\leftarrow$ w/a + c

we haven't yet defined what it means to be integrable on an unbounded set / open interval.

In short, we say f is (locally) integrable on $A \subseteq \mathbb{R}$ iff $\forall I$ compact $\subseteq \mathbb{R}$, $f|_I \in \mathcal{R}(I)$

[will cover properly in Improper Integrals]

• A Classic Caveat

We are used to thinking that differentiation & integration are "opposite" operations, in some sense. Thus if we were to differentiate a f^n , we would reasonably expect that we could integrate the resulting derivative to get back our original function. (up to a constant term)

This is NOT true. Let $f : [a, b] \rightarrow \mathbb{R}$ be differentiable. This does NOT guarantee $f' \in \mathcal{R}[a, b]$.

Eg.: $f(x) = x^2 \cos(x^{-2}) \mathbb{1}(x \in (0, 1])$ on $[0, 1]$

$$f'(x) = \begin{cases} 2x \cos(x^{-2}) + 2x^{-1} \sin(x^{-2}), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

f' is unbounded in a nbd. of 0

$\Rightarrow f' \notin \mathcal{R}[0, 1] \leftarrow \therefore \text{FTC doesn't apply!}$

(For this, we need something called the "generalized Riemann integral")

• Substitution Theorem / Change of Variables.

(B-5)

7.3.8 Substitution Theorem Let $J := [\alpha, \beta]$ and let $\varphi : J \rightarrow \mathbb{R}$ have a continuous derivative on J . If $f : I \rightarrow \mathbb{R}$ is continuous on an interval I containing $\varphi(J)$, then

$$(5) \quad \int_{\alpha}^{\beta} f(\varphi(t)) \cdot \varphi'(t) dt = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(x) dx.$$

The proof of this theorem is based on the Chain Rule 6.1.6, and will be outlined in Exercise 17. The hypotheses that f and φ' are continuous are restrictive, but are used to ensure the existence of the Riemann integral on the left side of (5).

essentially
change of
variables
works
as
expected.

Integration by Parts.

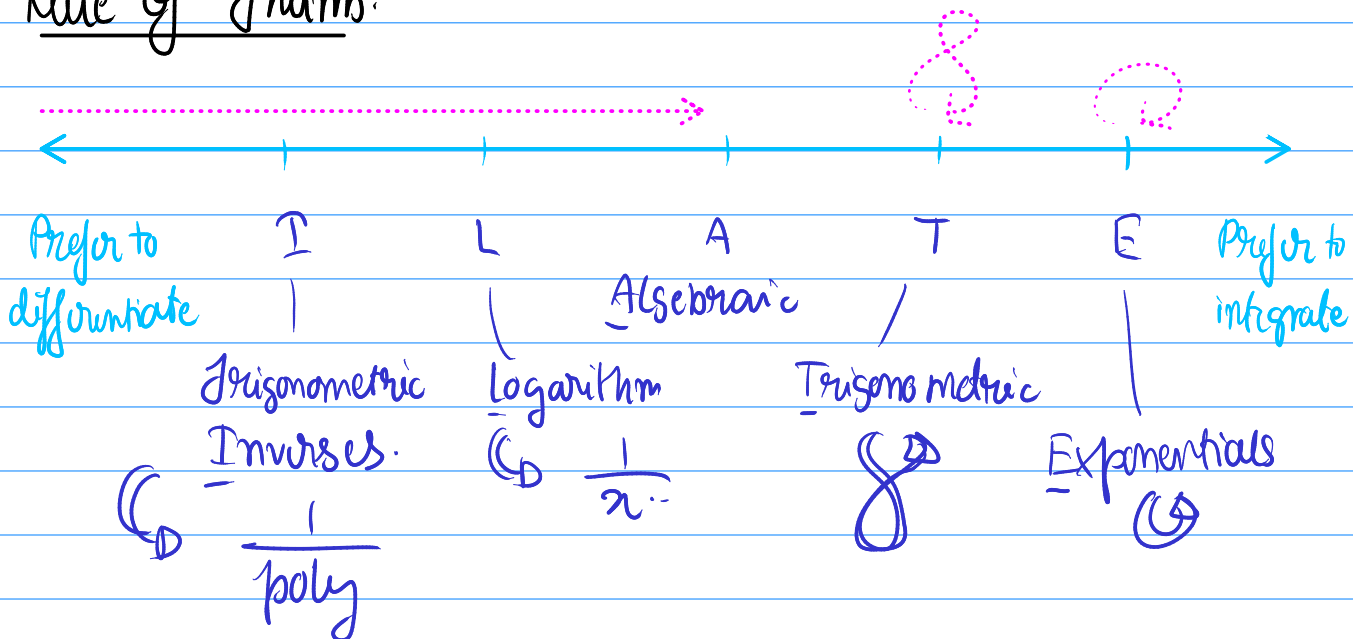
- $F, G : [a, b] \rightarrow \mathbb{R}$ differentiable; $F' = f, G' = g \in \mathcal{R}_{(a,b)}$

$$\therefore (FG)' = fG + Fg$$

$$\Rightarrow \int_a^b (FG)' = \int_a^b fG + \int_a^b Fg$$

$$\Rightarrow \int_a^b fG = [FG]_a^b - \int_a^b Fg$$

Rule of Thumb:



D-I Method:

$$fG \Rightarrow \begin{matrix} D \\ I \end{matrix}$$

$$\begin{matrix} + \\ G \end{matrix} \Rightarrow \begin{matrix} - \\ g \end{matrix} \quad \begin{matrix} + \\ f \end{matrix} \Rightarrow \begin{matrix} - \\ F \end{matrix}$$

Diagram illustrating the D-I Method. It shows the relationship between the function fG and its derivative $f'G + fG'$. The diagram shows fG being differentiated to get $f'G + fG'$, and then $f'G$ is integrated to get fG . The final result is $fG' + fG$.

②
(TOMATO)

i) Suppose f is cts. | $f(x) = \int_0^x f(t) dt$.
Prove $f \equiv 0$.

ii) Suppose $f: [1, \infty) \rightarrow \mathbb{R}$ differentiable & $f(1) = 1$.
Moreover:
$$f'(x) = \frac{1}{x^2 + f^2(x)}$$

ST: $f(x) \leq 1 + \frac{\pi}{4} \quad \forall x \geq 1$.

iii) Suppose f is a f^n | $f > 0$ & f' is cts. on $\mathbb{R}$.
If $f'(t) \geq (f(t))^{1/2}$, ST:

$$\sqrt{f(x)} \geq \sqrt{f(1)} + \frac{1}{2}(x-1) \quad \forall x \geq 1$$

iv) ST:

$$\int_0^\pi \left| \frac{\sin(mx)}{x} \right| dx \geq \frac{2}{\pi} \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} \right) \quad \text{--- } =: H_n$$

v) Find the area of region enclosed by the curves:

a) $y = x^2$

b) $x + y = 2$

c) $y = -\sqrt{x}$

slightly modified.

vi) [B. Math. 2008 Entrance] Let f be cts on $\mathbb{R}$

Suppose $f(x) = \frac{1}{t} \int_0^t [f(x+y) - f(y)] dy \quad \forall x \in \mathbb{R}, t > 0$

Then show f' exists & is a constant.

vii) Let $f(x)$ be C^2 on $[0, 2\pi]$ & $f''(x) > 0 \quad \forall x \in [0, 2\pi]$
ST: $\int_0^{2\pi} f(x) \cos(x) dx \geq 0.$

viii) Let f be a differentiable f^n | $f(f(x)) = x \quad \forall x \in [0, 1]$ involution
 Suppose $f(0) = 1$. Find:
 $\int_0^1 (x - f(x))^{2016} dx$

Ans: i)

$$f(x) = \int_0^x f(t) dt. \quad \Rightarrow \quad f(0) = \int_0^0 f(t) dt = 0.$$

$\therefore f$ is cts, $\int_0^x f(t) dt$ is differentiable $\Rightarrow f$ is differentiable

$$\therefore f'(x) = f(x)$$

$$\Rightarrow f(x) = e^x \quad \text{on} \quad \underline{f(x) = 0, \forall x}$$

$$\text{But } e^0 = 1 \neq 0$$

$\Rightarrow \Leftarrow$

Hence, proved.

Ans: ii)

$$f(1) = 1, \quad f(x) = \int_1^x f'(t) dt + f(1) \quad \left(\begin{array}{l} \text{Assuming} \\ f \in R[1, x] \\ \forall x > 1 \end{array} \right)$$

$$\therefore f(x) = 1 + \int_1^x \frac{dt}{t^2 + f^2(t)}$$

Check

$$\text{Now, } f'(1) = \frac{1}{1^2 + f^2(1)} = \frac{1}{2} > 0. \quad f'(x) > 0 \quad \forall x > 1$$

$$\Rightarrow f(x) > 1 \quad \forall x > 1$$

$$\therefore \frac{1}{t^2+1} \geq \frac{1}{t^2+f^2(t)}$$

$$\Rightarrow \int_1^x \frac{dt}{t^2+1} \geq \int_1^x \frac{dt}{t^2+f^2(t)}$$

$$\Rightarrow f(x) \leq 1 + \int_1^x \frac{dt}{t^2+1}$$

$$= 1 + \tan^{-1}(x) - \tan^{-1}(1)$$

$$= 1 + \tan^{-1}(x) - \pi/4$$

Now, $\tan^{-1}(x) \leq \pi/2$

$$\therefore f(x) \leq 1 + \pi/2 - \pi/4 = 1 + \pi/4$$

Ans. iii)

f is C^1 and $f > 0$.

Define $g(t) = \sqrt{f(t)}$. Applying LMVT on g on the interval $[1, x]$:

$$\frac{g(x) - g(1)}{x-1} = g'(\theta) \quad \text{where } \theta \in (1, x)$$

$$\parallel$$

$$\frac{\sqrt{f(x)} - \sqrt{f(1)}}{x-1}$$

Suffices to show:

$$g'(\theta) \geq \frac{1}{2}$$

$$\parallel$$

$$\frac{1}{2\sqrt{f(\theta)}} f'(\theta) \geq \frac{1}{2} \quad (\text{given})$$

Ans. iv)

$$\int_0^{\pi} \left| \frac{\sin(nx)}{n} \right| dx = \int_0^{\pi} \left| \frac{\sin(nx)}{nn} \right| n dx$$

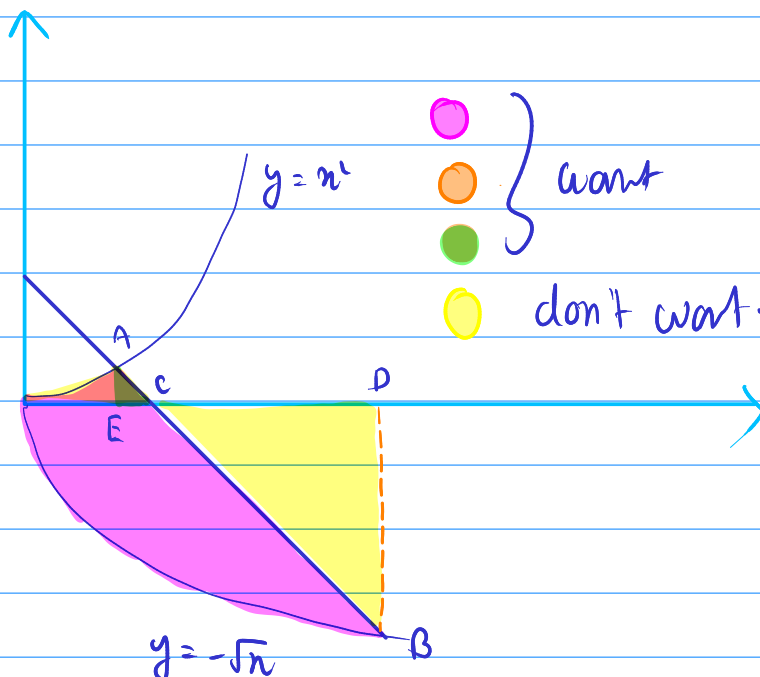
$$= \int_0^{n\pi} \left| \frac{\sin t}{t} \right| dt \quad [nx =: t, \quad n dx = dt]$$

$$= \sum_{j=1}^n \underbrace{\int_{(j-1)\pi}^{j\pi} \left| \frac{\sin t}{t} \right| dt}_{S_j} \geq \int_{(j-1)\pi}^{j\pi} \left| \frac{\sin t}{j\pi} \right| dt$$

$$= \frac{1}{j\pi} \int_0^{\pi} |\sin t| dt$$

$$\geq \sum_{j=1}^n \frac{2}{\pi j} = \frac{2}{\pi} \left(1 + \frac{1}{2} + \dots + \frac{1}{n} \right)$$

Ans. v)



Find the points
A, B, C, D, E.

Find relevant areas
& add.

Ans. vi) We know f cts on $\mathbb{R}$, hence locally integrable.

$$\frac{1}{t} \int_0^t [f(x+y) - f(y)] dy = f(x, t) \text{ is invariant int.}$$

Given any fixed x , $t f(x, t) = t f(x) = g_x(t)$

$$\therefore \frac{d}{dt} g_x(t) = f(x)$$

$$\parallel$$
$$\frac{d}{dt} \int_0^t [f(x+y) - f(y)] dy$$

$$= \frac{d}{dt} \left[\int_0^t f(x+y) dy - \int_0^t f(0+y) dy \right]$$

$$= \frac{d}{dt} \left[\int_x^{x+t} f(u) du - \int_0^t f(y) dy \right]$$

$$= f(x+t) - f(t) \quad \left[\begin{array}{l} \text{by FTC second form} \\ \text{(Leibniz Rule)} \end{array} \right]$$

$$\therefore \forall x, t: \quad f(x+t) - f(t) = f(x)$$

$$\Rightarrow f(x+t) = f(x) + f(t)$$

$$\Rightarrow f(x) = mx \text{ for some } m \in \mathbb{R}$$

$$\Rightarrow f'(x) = m, \text{ a constant.} \quad \blacksquare$$

Ans viii)

$$f(f(x)) = x, \quad f(0) = 1 \Leftrightarrow f(1) = 0$$

$$\text{Put } x = f(t) \text{ in } \int_0^1 (x - f(x))^{2016} dx$$

$$= \int_1^0 (f(t) - \cancel{f(f(t))}^t)^{2016} f'(t) dt$$

$$= - \int_0^1 (t - f(t))^{2016} f'(t) dt = I \text{ (let)}$$

$$\therefore I + I$$

$$= \int_0^1 (x - f(x))^{2016} (1 - f'(x)) dx$$

$$\underline{\text{Now}}, u = x - f(x), \quad du = (1 - f'(x)) dx$$

$$\begin{aligned} \therefore u(0) &= 0 - f(0) = -1 \\ u(1) &= 1 - f(1) = 1 \end{aligned}$$

$$\therefore 2I = \int_{-1}^1 u^{2016} du = 2 \int_0^1 u^{2016} du$$

$$\Rightarrow I = \frac{1}{2017}$$

★ Samojia ✓✓



Mean Value Theorems for Integrals

- First MVT for Integrals.

Suppose $f, g: [a, b] \rightarrow \mathbb{R}$. f cts. $g \geq 0$, $g \in \mathcal{R}[a, b]$

Then, $\exists c \in [a, b] \mid \int_a^b f(x) g(x) dx = f(c) \int_a^b g(x) dx$

- Second MVT for Integrals

(in our version, cts as well.)

Suppose $f, g: [a, b] \rightarrow \mathbb{R}$, f is non-decreasing & $g > 0$
 $g \in \mathcal{R}[a, b]$. Then $\exists \xi \in (a, b) \mid$

$$\int_a^b f(x) g(x) dx = f(a) \int_a^{\xi} g(x) dx + f(b) \int_{\xi}^b g(x) dx$$



Cauchy-Schwarz Inequality for Integrals

& equality hold iff

$$f(x) = \lambda g(x) \quad \forall x \in [a, b]$$

Let $f, g \in \mathcal{R}[a, b]$. Then:

$$\underbrace{\left[\int_a^b f(x) g(x) dx \right]^2}_B \leq \underbrace{\left(\int_a^b [f(x)]^2 dx \right)}_A \underbrace{\left(\int_a^b [g(x)]^2 dx \right)}_C$$

Proof: $h(t) := \int_a^b (f(x) - tg(x))^2 dx \geq 0 \quad \text{--- (1)}$

(1) $\Rightarrow \quad = At^2 - 2Bt + C \quad \text{where } A, B, C \dots$

$\therefore 4B^2 - 4AC \leq 0 \Rightarrow B^2 \leq AC$ equality holds when h has exactly 1 real root

☆ Improper Integrals

⊙ Motivation

- So far, we've dealt with Riemann integrals. However, they only work for bounded functions on compact intervals. What about unbounded f's / unbounded intervals / open intervals / any combination of these?
- We do what we've always done — try to take a limit

⊙ Introduction

- Defⁿ: We say f is locally integrable of $A \subseteq \mathbb{R}$ iff $\forall [a, b] \subseteq A$ compact in $\mathbb{R}$, $f \in \mathcal{R}[a, b]$.
- Eg: $\ln x$ is locally integrable on $(0, \infty)$,
 $\sin x$ is locally integrable on $\mathbb{R}$,
etc.
- Defⁿ: Let $f: [a, b) \rightarrow \mathbb{R}$ & f is locally integrable on $[a, b)$.
Then, we define $\int_a^b f(x) dx = \lim_{\theta \rightarrow b^-} \int_a^\theta f(x) dx$
This is called the improper integral of f on $[a, b)$
(similar for $(a, b]$, $[a, \infty)$, ...)
 $\lim_{\theta \rightarrow a^+} \int_\theta^b f(x) dx$, $\lim_{\theta \rightarrow \infty} \int_a^\theta f(x) dx$, ...
If the limit exists (& is finite), we say the improper integral converges.

- (a lot of language is similar to infinite series)
 - ✓ converges
 - ✓ diverges (to $\pm\infty$)
 - ✓ absolutely converges (absolutely integrable f^n)

⊙ Double Limit Situations $(a, b), (a, \infty), (-\infty, b), (-\infty, \infty)$

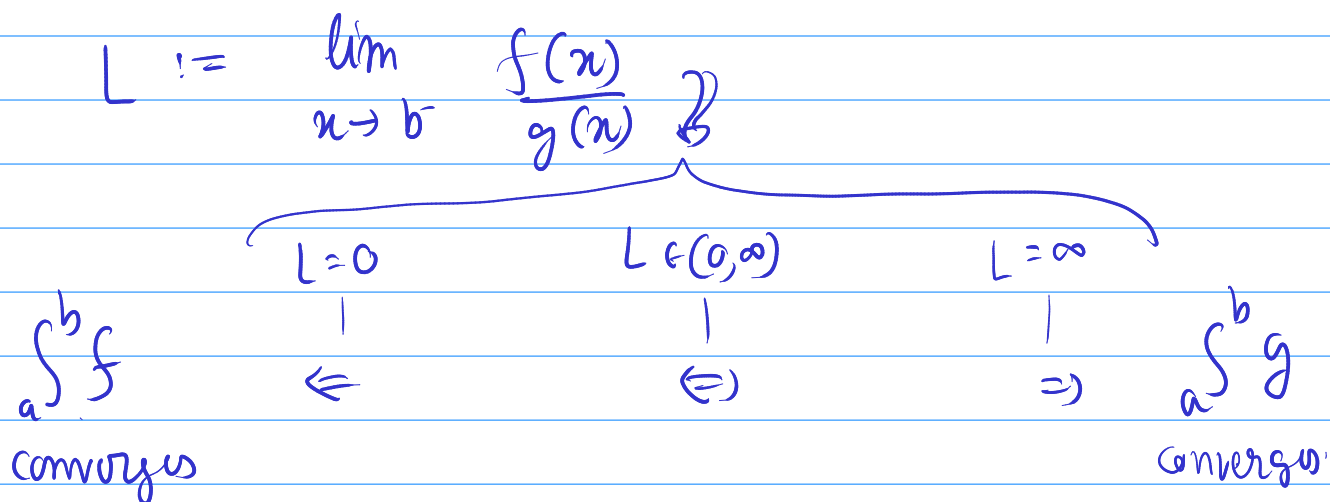
Extremely problematic: 2 limits, order matters etc
 & Cauchy Principal Value

⊙ Tests for Convergence

- Comparison Test (superseded by ↓)
- Limit Comparison Test (for $[a, b]$, others similar)

Let f, g be locally integrable on $[a, b]$

Assume $f \geq 0, g > 0$ on $[a, b]$



⊙ Some Common Improper Integrals

- $f(x) = x^{-p}$ on $(0, 1]$
$$\int_0^1 x^{-p} dx = \left. \frac{x^{1-p}}{1-p} \right|_0^1 \rightarrow \begin{cases} \frac{1}{1-p}, & \text{if } p < 1 \\ \infty, & \text{if } p > 1 \end{cases}$$

$$\int_0^1 x^{-1} dx = -\log 0 \rightarrow \infty$$

— else explodes to ∞
 $\therefore \int_0^1 x^{-p} dx$ converges to $\frac{1}{1-p}$ iff $p < 1$.

- $f(x) = x^{-p}$ on $[1, \infty)$

∴ (similar) (HW)

$$\int_1^{\infty} x^{-p} dx \text{ converges to } \frac{1}{p-1} \text{ iff } p > 1$$

- $\int_0^1 \ln x dx$ or $\int_{-\infty}^0 e^x dx$...

(HW)
same by graph reflections!

⊙ Beta & Gamma Integrals (@ Archishman)

- Remark: Almost all integral tricks (FTC, substitution, etc) work on improper integrals too!

END